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On the Statistical Dynamics of the Lorenz Model

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We use the theory of stochastic differential equations with rapidly fluctuating coefficients to study the statistical dynamics of the Lorenz model in the turbulent region. On the assumption that the system is ergodic we are able to calculate self-consistently several basic statistical quantities in terms of the parameters of the model. Our results are in good agreement with numerical computations.

KEY WORDS: Turbulence; Rayleigh–Bénard layer; random behavior of nonlinear differential equations; fluctuation spectra; stochastic differential equations.

1. INTRODUCTION

There are two basic approaches to the problem of turbulence. In the first, one seeks to obtain statistical solutions to the equations of motion by taking repeated moments of the equations and using some kind of a closure scheme to close the hierarchy of moment equations. In the second, one solves nonlinear differential equations obtained from the equations of motion, which have no explicit stochastic element in them, but which can, for certain ranges of values of the parameters, exhibit apparently random solutions owing to the appearance of a strange attractor. Both methods are assumed to be relevant to the problem of "turbulence," although the connection between the two is not immediately clear.

The present paper is an attempt to provide a connection between these two approaches. More specifically, we aim to show that it is possible to apply the techniques developed for the solution of stochastic differential equations (i.e., differential equations with randomly fluctuating coefficients) to systems

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of coupled nonlinear ordinary differential equations with strange attractors. We do this by exhibiting an example of such a calculation. We restrict ourselves here to the consideration of the Lorenz equations,[1] the first and most famous example of a system with a strange attractor. An analogous calculation for the Rössler equations[2] will be presented elsewhere. The Lorenz equations are derivable from the first nontrivial truncation of a modal expansion of the equations for Bénard convection in the Boussinesq approximation, and may be written in the standard dimensionless form

$$\dot{x} = \sigma(y - x) \tag{1a}$$
$$\dot{y} = rx - y - xz \tag{1b}$$
$$\dot{z} = -\nu z + xy \tag{1c}$$

Here the variable x measures the vertical convective velocity, y the temperature fluctuation, and z the mean convective temperature gradient; σ is the Prandtl number of the fluid, r is a reduced Rayleigh number ($r = 1$ for the onset of convection), and ν is related to the wave number of the convection rolls. If σ and ν are fixed at 10 and 8/3, respectively (the values originally used by Saltzman[3]), and r is gradually increased, it is found that at $r \simeq 24.74$ the solutions to the equations become unstable according to the linear theory, although there exist finite-amplitude instabilities already for $r > 21$. The system is then in the "turbulent" state, and remains in this state for $r \lesssim 250$. The model, based as it is on a physical system, is believed to describe qualitatively the onset of turbulence.[4]

In the following section we introduce our assumptions, and rewrite the Lorenz equations in a form suggestive of a stochastic differential equation. Applying certain results about this equation derived in the Appendix, we are then able to calculate several basic statistical moments of the solution in the turbulent regime in terms of the parameters of the model, and compare the results with the numerical evaluation of the corresponding quantities carried out by Lücke[5] and the author. A preliminary account of this work is given in Ref. 6. In view of some of the approximations made in solving the stochastic differential equation we find remarkably good agreement between the theory and the numerical results. Our conclusions are presented in Section 3.

2. STATISTICAL DYNAMICS OF THE LORENZ MODEL

In this section we shall be concerned with calculating various time averages, denoted by angular brackets, of the solution to Eqs. (1) in the turbulent regime. From the equations it is easy to show that

$$\langle x \rangle = \langle y \rangle = \langle xz \rangle = \langle yz \rangle = 0 \tag{2}$$

Throughout what follows, we shall assume that the solutions of (1) in the turbulent regime are ergodic. Thus we shall assume that we may identify averages over an ensemble of realizations of the solution with time averages in any one realization. Here a realization is specified by the initial conditions; if these lie outside the strange attractor, only that part of the solution that lies subsequently within the attractor is included. Moreover, stable periodic orbits within the turbulent regime are also excluded, comprising as they do a subset of all solutions that is of "zero measure." In particular we shall assume that the solutions (with the above qualification) are statistically stationary. There is ample evidence for this property from numerical investigation, but a rigorous mathematical proof of this property is not available. As a consequence of this assumption all time derivatives of averages vanish. By writing down quantities of the form

$$\frac{d}{dt}\langle A(x, y, z)\rangle = \left\langle \frac{\partial A}{\partial x}\dot{x} + \frac{\partial A}{\partial y}\dot{y} + \frac{\partial A}{\partial z}\dot{z}\right\rangle = 0 \tag{3}$$

and using Eqs. (1), it is possible to obtain an infinite number of relations between various averages. We obtain[5]

$$\langle x^k y\rangle = \langle x^{k+1}\rangle, \qquad k \text{ integer} \tag{4a}$$

$$\langle x^2\rangle = \nu\langle z\rangle \tag{4b}$$

$$\langle xyz\rangle = \nu\langle z^2\rangle \tag{4c}$$

$$\langle x^2 z\rangle = (\sigma + 1)(\langle y^2\rangle - \langle x^2\rangle) + \nu\langle z^2\rangle \tag{4d}$$

$$\langle x^2 z\rangle = \sigma(\langle y^2\rangle - \langle x^2\rangle) + (r - 1)\langle x^2\rangle \tag{4e}$$

$$(2\sigma + \nu)\langle x^2 z\rangle = 2\sigma\nu\langle z^2\rangle + \langle x^4\rangle \tag{4f}$$

These relations have been verified numerically,[5] providing further evidence for the validity of the stationariness hypothesis. From Eqs. (4) one obtains an important identity

$$\frac{1}{\sigma^2}\langle \dot{x}^2\rangle = \langle (y - x)^2\rangle = \nu[(r - 1)\langle z\rangle - \langle z^2\rangle] \tag{5}$$

An equivalent result has been given by Malkus[7]:

$$\langle x^2\rangle = \nu\langle z\rangle = \nu(r - 1) - \frac{1}{\sigma^2}\frac{\langle \dot{x}^2\rangle}{\langle z\rangle} - \frac{\langle (z - \langle z\rangle)^2\rangle}{\langle z\rangle} \tag{6}$$

Since $\langle x^2\rangle = \langle xy\rangle$ is the convective heat flux, this result shows that, within this model, the heat flux transported in turbulent convection has to be less than that transported in steady convection ($\dot{x} = 0, z = \langle z\rangle$). In particular, Eqs. (5) and (6) show that

$$(r - 1)^2 \geqslant (r - 1)\langle z\rangle \geqslant \langle z^2\rangle \tag{7}$$

with equality only in steady convection.

It may be observed that the number of relations of the type (3) is insufficient to determine all the averages. For example, the relations (4) enable one only to express the low-order moments in terms of the two unknown quantities $\langle z\rangle$ and $\langle z^2\rangle$, which are constrained only by the inequality (7). Our task will be to calculate these two moments.

We shall use the general method for solving stochastic differential equations with rapidly fluctuating coefficients.[8,9] Suppose that we have a stochastic differential equation of the form

$$df/dt = L(t)f \tag{8}$$

where $L(t)$ is a stochastic matrix. If L_0 is the mean of L, and $L_1 = L - L_0$ is the rapidly fluctuating part of L that need *not* be independent of f, then the mean of the process f, $\langle f\rangle$, satisfies the equation

$$\frac{d}{dt}\langle f\rangle = \left\{L_0 + \int_0^\infty d\tau\, \langle L_1(t)e^{L_0\tau}L_1(t-\tau)e^{-L_0\tau}\right\}\langle f(t)\rangle \tag{9}$$

In order to apply this theory to the Lorenz model we rewrite Eqs. (1), by eliminating y, in the form

$$\ddot{x} + b\dot{x} + [-a + \omega(t)]x = 0 \tag{10}$$

where

$$a = \sigma(r - 1 - \langle z\rangle), \qquad b = \sigma + 1 \tag{11}$$

$$\omega(t) = \sigma(z - \langle z\rangle) \tag{12}$$

and

$$\dot{z} + \nu z = \frac{d}{dt}\left(\frac{1}{2\sigma}x^2\right) + x^2 \tag{13}$$

Equation (10) is thus an equation of a linear "oscillator" with a zero-mean frequency modulation $\omega(t)$. The quantity a is always positive definite, as can be seen from the inequality (7). We know from numerical studies that in the turbulent regime the quantities x and z are both rapidly fluctuating in time. This can be seen, for example, in the paper by Robbins[10] dealing with the equations for the disk dynamo with a shunt, which can be transformed into the standard form (1). Thus $\omega(t)$ is a rapidly fluctuating zero-mean random process, and the approximation leading to Eq. (9) is valid.

The result (9) can be applied once Eq. (10) is written in the form (8). In view of (2), the first nontrivial moments of x and $\dot{x}$ are quadratic ones. As shown in the Appendix, Eq. (9) then has statistically stationary solutions provided

$$\alpha = a\lambda^2/b \tag{14}$$

Here $\lambda^2 = 4a + b^2$, and

$$\alpha = \int_0^\infty d\tau\, \eta(\tau) \cosh \lambda\tau \tag{15}$$

where $\eta(\tau)$ is the autocorrelation function of the process $\omega(t)$, defined by

$$\langle \omega(t)\omega(t')\rangle \equiv \eta(t - t') \equiv \eta(\tau) \tag{16}$$

In what follows we shall assume that $\omega(t)$ has an autocorrelation function that falls off sufficiently rapidly for the quantity α to be well defined. The condition (14) gives the "strength" of the fluctuations, or the energy input, required on average to counterbalance the damping term, and is thus an example of a fluctuation-dissipation theorem for an equilibrium system. The derivation leading to the result (14) requires that the autocorrelation time of $\omega(t)$ be short, but not too short, an assumption that is verified a posteriori, and that is in agreement with the numerical results.[10] A δ-function-correlated process would be unable to balance the dissipation. The corresponding stationary solution to Eq. (9) is

$$\langle \dot{x}^2\rangle = a\langle x^2\rangle \tag{17}$$

this being the analog for our oscillator of the usual virial theorem.

We now apply these results to the calculation of the quantities $\langle z\rangle$ and $\langle z^2\rangle$. Consider first the quantity $\Omega_0{}^2 \equiv \langle \dot{x}^2\rangle/\langle x^2\rangle$. From Eqs. (11) and (17) we obtain

$$\Omega_0{}^2 = \sigma(r - 1 - \langle z\rangle) \tag{18}$$

On the other hand, using Eq. (6), we find

$$\Omega_0{}^2 = \sigma^2\left(r - 1 - \frac{\langle z^2\rangle}{\langle z\rangle}\right) \tag{19}$$

We shall be interested in studying the quantities ζ and Δ defined by

$$\langle z\rangle = (r - 1)(1 - \zeta), \qquad 0 < \zeta < 1 \tag{20}$$

$$\langle z^2\rangle = \langle z\rangle^2/(1 - \Delta^2), \qquad 0 < \Delta < 1 \tag{21}$$

From Eqs. (18)–(21) we then obtain the relation

$$\zeta = \sigma\Delta^2/(\sigma - 1 + \Delta^2) \tag{22}$$

We have carried out, following in detail the method of Lücke,[5] a numerical calculation of the quantities ζ and Δ as a function of r for $\sigma = 10$ and $\nu = 8/3$. Our results, reproduced in Fig. 1, are in agreement with those of Lücke, in spite of the fact that we have taken as our initial conditions $x = y = z = 1$ rather than the r-dependent initial conditions chosen by Lücke. This agreement supports our hypothesis that the solution to the Lorenz equations in the turbulent regime is indeed ergodic. As seen from

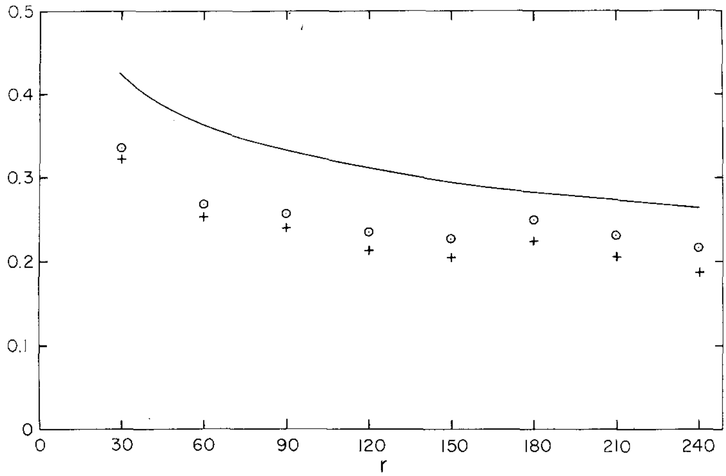

Fig. 1. Numerical results for $\zeta^{1/2}$ (circles) and Δ (crosses) as a function of r, for $\sigma = 10$ and $\nu = 8/3$. Also shown is the theoretical result [Eq. (33)] for Δ (line).

Fig. 1, the relationship between the curves of $\zeta^{1/2}$ and Δ as functions of r is in excellent agreement with the prediction (22), particularly for $r < 100$. For larger values of r, the "constant" of proportionality $\sigma^{1/2}/(\sigma - 1 + \Delta^2)^{1/2}$ increases with r as indicated by its Δ dependence, although at a somewhat faster rate. As a further check on the result (22) we have plotted in Fig. 2 the

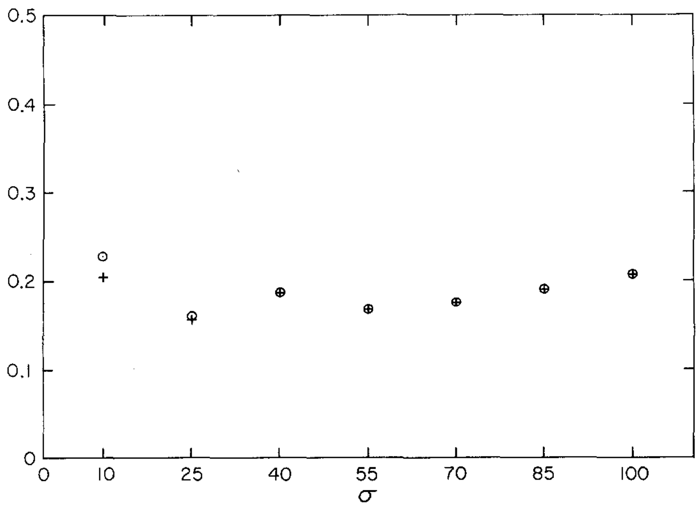

Fig. 2. Numerical results for $\zeta^{1/2}$ (circles) and Δ (crosses) as a function of σ, for $r = 150$ and $\nu = 8/3$.

results of a numerical calculation of the functional dependence of $\zeta^{1/2}$ and Δ on σ at $r = 150$, for values of σ satisfying $10 < \sigma < 100$. The value of r chosen ensures that the system is in the turbulent regime for all the values of σ considered. The numerical results show that although $\zeta^{1/2}$ exceeds Δ for smaller values of σ, it becomes essentially indistinguishable from it at larger values of σ, in complete agreement with the relation (22).

We now introduce the autocorrelation time τ_c of the process $\omega(t)$ by the relation $\alpha \equiv \eta(0)\tau_c$. Substituting from Eqs. (12) and (14)–(16), we obtain

$$\zeta^2(r-1)^2 + \frac{1}{4}\frac{(\sigma+1)^2}{\sigma}\zeta(r-1) = \frac{1}{4}\tau_c(\sigma+1)\frac{\Delta^2}{1-\Delta^2}(r-1)^2(1-\zeta)^2 \tag{23}$$

or, using Eq. (22) to eliminate ζ,

$$\Delta^2 = (\sigma^2 - 1)\frac{\tau_c(r-1)(\sigma-1) - (\sigma+1)}{4\sigma^2(r-1) + \tau_c(r-1)(\sigma^2-1)(\sigma-1) + (\sigma+1)^2} \tag{24}$$

Suppose that, as a first approximation, we take Δ to be independent of r. Then ζ is also independent of r, and it is necessary that $\tau_c r \gg 1$, with τ_c independent of r. Hence

$$\zeta \simeq \tau_c(\sigma^2 - 1)/[4\sigma + \tau_c(\sigma^2 - 1)] \tag{25}$$

From the numerical results in Fig. 1 it follows that $\tau_c \simeq 0.03$. Observe that the dimensionless correlation time is a small number, so that the short autocorrelation time approximation required for the derivation of Eq. (9) is indeed valid. A more detailed discussion of the value of τ_c and its r and σ dependence is given below.

Thus far we have only used the equation for x in terms z. In order to calculate ζ and Δ, we now use Eq. (13) for z in terms of x and calculate approximately but self-consistently the quantity τ_c. Equation (13) is essentially a Langevin equation, and can be written in the form

$$\dot{z}' + \nu z' = A(t) \equiv (1/\sigma)x\dot{x} + x^2 - \nu\langle z\rangle \tag{26}$$

where $z' = z - \langle z\rangle$, and $A(t)$ is a zero-mean, rapidly fluctuating forcing term. From the formal solution to Eq. (26), we find that

$$\begin{aligned}\langle(z')^2\rangle &= \int_0^t\int_0^t dt'\,dt''\exp[-\nu(t-t')]\exp[-\nu(t-t'')]\\ &\quad\times\langle A(t')A(t'')\rangle \simeq (1/2\nu)\tau_c'\langle A^2\rangle\end{aligned} \tag{27}$$

where we have assumed that the forcing term has a short autocorrelation time τ_c', and that the process z' is statistically stationary. Thus

$$\langle z^2\rangle - \langle z\rangle^2 = (1/2\nu)\tau_c'(\langle x^2y^2\rangle - \nu^2\langle z\rangle^2) \tag{28}$$

In order to obtain a closed equation for τ_c we shall assume that the correlation times τ_c and τ_c' are comparable. It is shown in the Appendix that under statistically stationary conditions the following result can be derived from Eq. (10):

$$a\langle x^4\rangle = \langle \dot{x}^2 x^2\rangle \tag{29}$$

Using Eqs. (1) and (4), we can write this result in the form

$$\langle x^2y^2\rangle = \nu\left(1+\frac{a}{\sigma^2}\right)[(\sigma\nu+2\sigma^2+2\sigma+\nu)(r-1)\langle z\rangle - \sigma(\nu+2\sigma+2)\langle z^2\rangle] \tag{30}$$

Using the definitions (20) and (21), we see that it now follows from the results (28) and (30) that

$$2\Delta^2 + \tau_c\nu(1-\Delta^2) = \tau_c\left(1+\frac{(r-1)\zeta}{\sigma}\right)\left\{\nu + \frac{(2\sigma+\nu)(\sigma+1)(\sigma^2-1)\tau_c}{4\sigma^2+(\sigma^2-1)(\sigma-1)\tau_c}\right\} \tag{31}$$

Upon using the results (22) and (24), we obtain an algebraic equation for τ_c. In order to calculate τ_c it is convenient to make use of the approximations $\tau_c\sigma \ll 4$ and $\tau_c r \gg 1$. With these approximations, it can be shown that

$$\tau_c \simeq 2\left(\frac{\sigma-1}{\sigma+1}\right)^{1/2}\left(\sigma+\frac{1}{2}\nu\right)^{-1/2}(r-1)^{-1/2} = 0.54(r-1)^{-1/2} \tag{32}$$

for $\sigma = 10$, $\nu = 8/3$. We see now that the correlation time does have a weak Rayleigh-number dependence; it varies between $0.10 > \tau_c > 0.03$ for $50 < r < 240$. This is in satisfactory agreement with the simpler estimate made above. With the result (12), we now have

$$\Delta \simeq \left\{\frac{(\sigma^2-1)(\sigma-1)\tau_c}{4\sigma^2+(\sigma^2-1)(\sigma-1)\tau_c}\right\}^{1/2} \simeq 1.04(r-1)^{-1/4} \tag{33}$$

The theoretical result (33) is also plotted in Fig. 1. We see that the theory correctly predicts the r dependence, but gives the constant of proportionality approximately 35% too large. This discrepancy is not surprising in view of the very approximate nature of the result (32). In particular the approximation $\tau_c' \sim \tau_c$ used in its derivation has to be regarded with some caution; a different constant of proportionality between τ_c' and τ_c would not alter the r dependence, but would alter the numerical coefficient in (32).

As a further check on the results (32) and (33), we observe from Fig. 2 that Δ increases in proportion to $\sigma^{1/4}$, as predicted by the theory, although the coefficient of proportionality is again too large.

Lücke[5] has reported numerical results for several fundamental frequencies of the Lorenz model as a function of r. In the remainder of this section we obtain the predictions of the present theory for two of them, and compare them with the numerical results. Consider first the quantity Ω_0. From Eqs. (18), (20), and (25) we obtain, for large r,

$$\Omega_0^2 \equiv \sigma(r-1)\zeta = \frac{\sigma(\sigma^2-1)\tau_c}{4\sigma+(\sigma^2-1)\tau_c}(r-1) \tag{34}$$

Lücke has suggested that, numerically, Ω_0 is well approximated by $\Omega_0 \sim (r-1)^{1/2}$. This result would be in agreement with the present theory with the approximation of ζ (or τ_c) by a suitable constant ($\tau_c \sim 0.045$). On the other hand, we have seen that τ_c is not really a constant, but decreases with r. This does not seem to be in such good agreement with the numerical result, although there may be some evidence at large r for a slower increase with r than $r^{1/2}$. Lücke has also considered the quantity $\Omega_z^2 \equiv \langle \dot{z}^2\rangle / \langle (z-\langle z\rangle)^2\rangle$. From Eqs. (1), (4), (21), and (30) we find that Ω_z is given by

$$\Omega_z^2 = -\frac{\nu}{\Delta^2}\left[\nu + \left(1+\frac{a}{\sigma^2}\right)\sigma(2\sigma+\nu+2)\right] + \frac{1-\Delta^2}{\Delta^2}\left(1+\frac{a}{\sigma^2}\right)\frac{\nu(r-1)}{\langle z\rangle}(\nu+2\sigma^2+\sigma\nu+2\sigma) \tag{35}$$

Substituting for a and $\langle z\rangle$ from Eqs. (11) and (20), and using the result (22) to eliminate ζ, we obtain

$$\Omega_z^2 = \frac{\nu}{\sigma-1}(2\sigma^2+2\sigma+\nu\sigma+\nu) + \frac{\nu}{\sigma-1+\Delta^2}\left[\nu+\frac{\Delta^2}{\sigma-1}(2\sigma^2+2\sigma+\nu\sigma+\nu)\right](r-1) \tag{36}$$

Setting $\Delta^2 = 0.09$, the value that reproduces Lücke's result for Ω_0, we obtain

$$\Omega_z^2 = 73.88 + 1.52(r-1) \tag{37}$$

This is in reasonable agreement with the numerical result $\Omega_z \sim 1.77(r-1)^{1/2}$. In particular we see that for small r ($r = 30$) the predicted value of Ω_z ($\Omega_z = 10.86$) agrees well with that computed both numerically and using a simple mode coupling approximation,[(5)] while for large r the $r^{1/2}$ dependence is recovered. Unlike the case of Ω_0, the $r^{1/2}$ dependence persists for large r even when the r dependence of Δ is incorporated. We have also computed the quantity Ω_z as a function of σ for $r = 150$. The comparison of the results with the present theory is shown in Fig. 3, where we have plotted the quantity Ω_z as given by Eq. (36), with $\Delta^2 = 0.09(\sigma/10)^{1/2}$. This choice of Δ incorporates approximately the $\sigma^{1/4}$ behavior shown in Eq. (33). We see that the theory is again in reasonably good agreement with the numerical results, although the predicted σ dependence is somewhat stronger. We have not calculated Ω_0 as a function of σ, because the numerical technique was in-

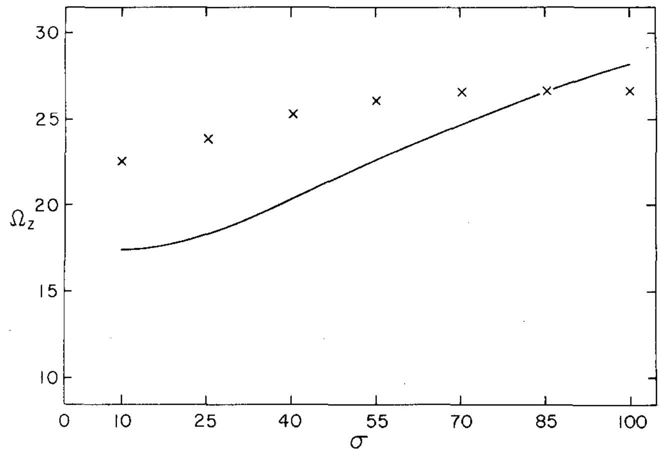

Fig. 3. Numerical results for Ω_z as a function of σ, for $r = 150$ and $\nu = 8/3$. The line is the theoretical result (36) with $\Delta^2 = 0.09(\sigma/10)^{1/2}$.

sufficiently accurate for large σ, when $\zeta^{1/2}$ and Δ are essentially indistinguishable. On the basis of the above results we may conclude that the dependence of the coefficient in the empirical result $\Omega_z \sim 1.77(r - 1)^{1/2}$ on the parameters ν and σ suggested by Lücke[5] cannot be correct.

Finally, we note that there appears no way of calculating the quantity $\Omega_\infty{}^2 \equiv \langle \ddot{x}^2 \rangle / \langle \dot{x}^2 \rangle$, also computed by Lücke,[5] using only the fourth-order moments used above.

3. DISCUSSION

In this paper we have seen that some systems of differential equations with strange attractors, such as the Lorenz model, can be treated by standard statistical methods that are used in treating "noisy" systems. We have seen that these methods predict the correct functional dependence of certain statistical averages on the Rayleigh number r, as well as giving the correct amplitude to a good accuracy. In particular, we have found that the relationship between the quantities ζ and Δ^2 describing the deviation of $\langle z \rangle$ and $\langle z^2 \rangle$ from the steady-state convection solution is predicted very accurately. We have also seen that the autocorrelation time τ_c has a weak r and σ dependence, but that it remains small, thereby justifying the use of the short-autocorrelation-time approximation. In deriving the identities (17) and (29) between certain statistical moments in the stationary state we have made use of further assumptions in order to simplify the calculation (see Appendix). As a consequence the predictions of the behavior of the basic frequencies Ω_0 and Ω_z

are quantitatively not as accurate as they might otherwise be. The theory predicts correctly the general dependence on r, although the coefficients are accurate to only 25%. Similarly, the self-consistent calculation of the autocorrelation time predicts the correct r dependence, but with an amplitude somewhat too large. We would expect that an even better agreement with the numerical results could be achieved if a more careful discussion were undertaken.

In this way we have shown that strange attractors and "noisy" systems, while apparently dissimilar, can have a great deal in common if we are only interested in their statistical properties. Indeed, it is likely that the origin of many stochastic systems lies in hidden nonlinear systems with strange attractors, and it is therefore reassuring to know that the details of the process producing the fluctuations are irrelevant for the gross properties of the system.

APPENDIX

In this Appendix we present the details of the calculation of the second and fourth moments of the solution to the stochastic differential equation (10) in the short-autocorrelation-time approximation (9). To the author's knowledge, the present method has not been applied to a damped simple harmonic oscillator with a random frequency component. The only treatment of such a system hitherto carried out has been by Bourret[11] using the Bourret integral equation.[12] However, this equation is not a self-consistent approximation for short autocorrelation times.[8]

From Eq. (10) it is easy to obtain an equation for the quadratic moments of x and $\dot{x}$ in the form (8), with

$$f = \begin{pmatrix} x^2 \\ x\dot{x} \\ \dot{x}^2 \end{pmatrix}, \qquad L_0 = \begin{pmatrix} 0 & 2 & 0 \\ a & -b & 1 \\ 0 & 2a & -2b \end{pmatrix}, \qquad L_1 = \begin{pmatrix} 0 & 0 & 0 \\ -\omega & 0 & 0 \\ 0 & -2\omega & 0 \end{pmatrix} \tag{A1}$$

In order to apply Eq. (9), we have to calculate the quantity $\exp L_0\tau$. This can be done most easily in the following way. Observe that $f = f_0 \exp L_0\tau$ is a solution of the set of ordinary differential equations

$$\dot{f} = L_0 f \tag{A2}$$

Seeking solutions proportional to $\exp st$, we can calculate the eigenvalues s of the system (A2). They are given by

$$s = -b, \qquad s = -b \pm \lambda, \qquad \lambda^2 = 4a + b^2 \tag{A3}$$

The general solution to the system (A2) is then a superposition of these three fundamental solutions:

$$f_1 = e^{-bt}(A + Be^{\lambda t} + Ce^{-\lambda t}) \tag{A4a}$$

The corresponding expressions for f_2 and f_3 now follow from Eq. (A2):

$$f_2 = \tfrac{1}{2}e^{-bt}[-bA + (\lambda - b)Be^{\lambda t} - (\lambda + b)Ce^{-\lambda t}] \tag{A4b}$$

$$f_3 = \tfrac{1}{2}e^{-bt}[-2aA + (\lambda^2 - b\lambda - a)Be^{\lambda t} + (\lambda^2 + b\lambda - a)Ce^{-\lambda t}] \tag{A4c}$$

If the coefficients A, B, and C are now eliminated in favor of $f_1(0)$, $f_2(0)$, and $f_3(0)$, Eqs. (A4) can be written in the form $f_i(t) = S_{ij}f_j(0)$, where $S = \exp L_0 t$ is now known. The following elements of S will be required in what follows:

$$\lambda^2 S_{11} = e^{-bt}(2a + b\lambda \sinh \lambda t + (2a + b^2)\cosh \lambda t) \tag{A5a}$$

$$\lambda^2 S_{12} = 2e^{-bt}(-b + \lambda \sinh \lambda t + b \cosh \lambda t) \tag{A5b}$$

$$\lambda^2 S_{13} = 2e^{-bt}(\cosh \lambda t - 1) \tag{A5c}$$

$$\lambda^2 S_{21} = e^{-bt}[ab(\cosh \lambda t - 1) + a\lambda \sinh \lambda t] \tag{A5d}$$

$$\lambda^2 S_{22} = e^{-bt}(b^2 + 4a \cosh \lambda t) \tag{A5e}$$

$$\lambda^2 S_{23} = e^{-bt}[-b(\cosh \lambda t - 1) + \lambda \sinh \lambda t] \tag{A5f}$$

The quantity $\exp -L_0 t$ is obtained by changing the sign of t in the above expressions. Evaluating the expression on the right side of Eq. (9) using the results (A1) and (A5), we obtain finally the equations

$$(d/dt)\langle x^2 \rangle = 2\langle x\dot{x} \rangle \tag{A6a}$$

$$\begin{aligned}(d/dt)\langle x\dot{x} \rangle &= \{a + (2/\lambda^2)[b(\gamma - \alpha) + \lambda\beta]\}\langle x^2 \rangle \\ &\quad + [-b + (4/\lambda^2)(\gamma - \alpha)]\langle x\dot{x} \rangle + \langle \dot{x}^2 \rangle\end{aligned} \tag{A6b}$$

$$\begin{aligned}(d/dt)\langle \dot{x}^2 \rangle &= (4/\lambda^2)[a\gamma + (a + \tfrac{1}{2}b^2)\alpha - \tfrac{1}{2}b\lambda\beta]\langle x^2 \rangle \\ &\quad + 2a\langle x\dot{x} \rangle + [-2b + (4/\lambda^2)(\gamma - \alpha)]\langle \dot{x}^2 \rangle\end{aligned} \tag{A6c}$$

where

$$\alpha = \int_0^\infty d\tau\, \eta(\tau) \cosh \lambda\tau, \qquad \beta = \int_0^\infty d\tau\, \eta(\tau) \sinh \lambda\tau$$

$$\gamma = \int_0^\infty d\tau\, \eta(\tau) \tag{A7}$$

and $\eta(\tau)$ is the autocorrelation function of the process $\omega(t)$ defined by Eq. (16). We have again assumed that the process $\omega(t)$ is statistically stationary, and that the quantity α is well defined.

By assumption, the process x is also stationary, so that we may set the time derivatives of all averages equal to zero. From Eqs. (A6b) and (A6c) it then follows that

$$\begin{aligned}&\{a + (2/\lambda^2)[b(\gamma - a) + \lambda\beta]\}[b - (2/\lambda^2)(\gamma - \alpha)] \\ &\quad + (1/\lambda^2)[2a\gamma + (2a + b^2)\alpha - b\lambda\beta] = 0\end{aligned} \tag{A8}$$

An examination of the definitions (A7) suggests the approximation

$$\beta, \gamma \ll \alpha \tag{A9}$$

valid for correlation times that are short, but not too short, as observed in the numerical results.[10] It follows that the condition (14) is required for statistically stationary solutions. On substituting this condition into Eq. (A6b), we obtain the result (17).

We now turn to the fourth-order moments. Writing the equations for the five fourth-order moments in the form (8), we obtain

$$f \equiv \begin{pmatrix} x^4 \\ \dot{x}x^3 \\ \dot{x}^2x^2 \\ \dot{x}^3x \\ \dot{x}^4 \end{pmatrix}, \qquad L_0 \equiv \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ a & -b & 3 & 0 & 0 \\ 0 & 2a & -2b & 2 & 0 \\ 0 & 0 & 3a & -3b & 1 \\ 0 & 0 & 0 & 4a & -4b \end{pmatrix} \tag{A10a}$$

and

$$L_1 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -\omega & 0 & 0 & 0 & 0 \\ 0 & -2\omega & 0 & 0 & 0 \\ 0 & 0 & -3\omega & 0 & 0 \\ 0 & 0 & 0 & -4\omega & 0 \end{pmatrix} \tag{A10b}$$

Again we have to calculate the matrix $\exp L_0\tau$. Proceeding as before, we find that the eigenvalues s of L_0 are given by

$$s = -2b,\ -2b \pm \lambda,\ -2b \pm 2\lambda \tag{A11}$$

where again $\lambda^2 = 4a + b^2$. The general solution is thus

$$f_1(t) = e^{-2bt}(A + Be^{\lambda t} + Ce^{-\lambda t} + De^{2\lambda t} + Ee^{-2\lambda t}) \tag{A12}$$

with corresponding results for $f_2(t),..., f_5(t)$ obtained from the equation $\dot{f} = L_0 f$. We can write the result in the form

$$f(t) = e^{-2bt}T\{A, Be^{\lambda t}, Ce^{-\lambda t}, De^{2\lambda t}, Ee^{-2\lambda t}\} \tag{A13}$$

where the matrix T has the rows

$$T_{1j} = (1, 1, 1, 1, 1) \tag{A14a}$$

$$T_{2j} = (-\tfrac{1}{2}b,\ \pm\tfrac{1}{4}\lambda - \tfrac{1}{2}b,\ \pm\tfrac{1}{2}\lambda - \tfrac{1}{2}b) \tag{A14b}$$

$$T_{3j} = (-\tfrac{1}{3}a + \tfrac{1}{6}b^2,\ \mp\tfrac{1}{4}\lambda b + \tfrac{1}{4}b^2,\ a \mp \tfrac{1}{2}\lambda b + \tfrac{1}{2}b^2) \tag{A14c}$$

$$T_{4j} = (\tfrac{1}{2}ab,\ \mp\tfrac{1}{4}a\lambda \pm \tfrac{1}{8}b^2\lambda - \tfrac{1}{8}b^3,\ -\tfrac{3}{2}ab \pm \tfrac{1}{2}\lambda \pm \tfrac{1}{2}b^2\lambda - \tfrac{1}{2}b^3) \tag{A14d}$$

$$T_{5j} = (a^2,\ -a^2 - \tfrac{1}{2}ab^2 \pm \tfrac{1}{2}ab\lambda,\ a^2 + 2ab - \tfrac{1}{2}b^4 \mp \tfrac{1}{2}b^3\lambda \mp ab\lambda) \tag{A14e}$$

We can now determine the constants A, B, C, D, and E in terms of $f_i(0)$ ($i = 1,\dots, 5$). We obtain

$$\lambda^4 A = 6a^2 f_1(0) - 12ab f_2(0) + (6b^2 - 12a) f_3(0) + 12b f_4(0) + 6 f_5(0) \quad \text{(A15a)}$$

$$\begin{aligned}\lambda^4 B = {} & (4a^2 + 2ab^2 + 2ab\lambda) f_1(0) + (-2b^3 + 4a\lambda - 2\lambda b^2) f_2(0) \\ & + (-6b^2 - 6b\lambda) f_3(0) + (-4\lambda - 8b) f_4(0) - 4 f_5(0)\end{aligned} \quad \text{(A15b)}$$

$$\begin{aligned}\lambda^4 D = {} & (a^2 + 2ab + \tfrac{1}{2}b^4 + \tfrac{1}{2}b^3\lambda + ab\lambda) f_1(0) + (6ab + 2b^3 + 2a\lambda + 2\lambda b^2) f_2(0) \\ & + (3b^2 + 6a + 3b\lambda) f_3(0) + (2b + 2\lambda) f_4(0) + f_5(0)\end{aligned} \quad \text{(A15c)}$$

$$C(\lambda) = B(-\lambda), \qquad E(\lambda) = D(-\lambda) \quad \text{(A15d)}$$

The above relations determine the matrix $\exp L_0 t$ as the matrix of coefficients of $f_i(0)$ in the equations for $f_i(t)$. In order to calculate the right-hand side of Eq. (9), let

$$\langle L_1(t)(\exp L_0\tau) L_1(t - \tau) \exp - L_0\tau\rangle = \eta(\tau)\gamma(\tau) \quad \text{(A16)}$$

where γ is a 5×5 matrix, and η the autocorrelation function of the process $\omega(t)$. We shall be interested in the correlations $\langle x^4\rangle$ and $\langle \dot{x}^2 x^2\rangle$. From Eq. (9) we find that these are connected by the equations

$$\frac{d}{dt}\langle x^4\rangle = 4\langle \dot{x}x^3\rangle \quad \text{(A17a)}$$

$$\begin{aligned}\frac{d}{dt}\langle \dot{x}x^3\rangle = {} & \left[a + \int_0^\infty d\tau\, \eta\gamma_{21}(\tau)\right]\langle x^4\rangle + \left[-b + \int_0^\infty d\tau\, \eta\gamma_{22}(\tau)\right]\langle \dot{x}x^3\rangle \\ & + \left[3 + \int_0^\infty d\tau\, \eta\gamma_{23}(\tau)\right]\langle \dot{x}^2 x^2\rangle\end{aligned} \quad \text{(A17b)}$$

After a considerable amount of algebra the expressions for γ_{21} and γ_{23} reduce to

$$\gamma_{21} = -(4b/\lambda^2)(\cosh \lambda\tau - 1) + (4/\lambda)\sinh \lambda\tau, \qquad \gamma_{23} = 0 \quad \text{(A18)}$$

In a statistically stationary state γ_{22} will not be required. Equations (A17) now give, using again the approximation (A9), the relation

$$[(4b\alpha/\lambda^2) - a]\langle x^4\rangle = 3\langle \dot{x}^2 x^2\rangle \quad \text{(A19)}$$

Finally, using the stationariness condition (14), we find that the relation (A19) reduces to the result (29).

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